1. Integrate the following

(a) $\int \cos^5 x \sin x \, dx = -\frac{1}{6} \cos x + c$

(b) $\int \frac{x-1}{x^2-1} \, dx = \int \frac{1}{x+1} \, dx = \ln|x+1| + c$

(c) $\int \frac{x+3}{x^2-1} \, dx = \int \left(\frac{-1}{x+1} + \frac{2}{x-1} \right) \, dx = -1 \ln|x+1| + 2\ln|x-1| + c$

(d) $\int \sqrt{x} \ln x \, dx = \frac{2}{3} x^{3/2} \ln x - \int \frac{2\sqrt{x}}{3} \, dx = \frac{2}{3} x^{3/2} \ln x - \frac{4}{9} x^{3/2} + c$

(e) $\int \frac{x}{\sqrt{2+3x^2}} \, dx = \frac{1}{3} \sqrt{(2+3x^2)} + c$

(f) $\int x \cos x \, dx = x \sin x + \cos x + c$

2. Evaluate the definite integrals:

(a) $\int_3^4 \frac{5x-2}{x^2-4} \, dx = \int_3^4 \left(\frac{2}{x-2} + \frac{3}{x+2} \right) \, dx = (2\ln|x-2| + 3\ln|x+2|)|_3^4 = \ln \frac{864}{125}$

(b) $\int_1^{e^2} \frac{\ln^2 x}{x} \, dx = \frac{1}{3} \ln^3 x \Big|_1^{e^2} = \frac{8}{3}$

(c) $\int_0^{1/2} \frac{1}{1-x^2} \, dx = \int_0^{1/2} \frac{1}{2} \left(\frac{1}{1-x} + \frac{1}{1+x} \right) \, dx$
$= \frac{1}{2} \left(-\ln(1-x) + \ln(1+x) \right) \Big|_0^{1/2} = \frac{1}{2} \ln 3$

(d) $\int_0^1 x^3 \ln x \, dx = \left(\frac{1}{4} x^4 \ln x - \frac{1}{16} x^4 \right) \Big|_{t \to 0}^1 = (0 - \frac{1}{16}) - (0 - 0) = -\frac{1}{16}$

(e) $\int_0^1 x\sqrt{1+x^2} \, dx = \frac{1}{3}(1+x^2)^{3/2} \Big|_0^1 = \frac{1}{3}(2^{3/2} - 1)$

(f) $\int_0^\infty t e^{-t} \, dt = \left(-t e^{-t} - e^{-t} \right) \Big|_0^{t=x \to \infty} = 1$

3. Solve the initial value problems:

(a) $\frac{dy}{dx} = 2x\sqrt{1-y^2}$ $\qquad y(0) = 1/2.$

answer: $y(x) = \sin \left(x^2 + \frac{\pi}{6} \right)$

(b) $\dfrac{dy}{dx} = x + xy^2 \qquad\qquad y(0) = 1$

answer: $y(x) = \tan\left(\frac{1}{2}x^2 + \frac{\pi}{4}\right)$

4. Use the washer method to calculate the volume of the solid of revolution that is generated by revolving the region bounded by the curves $y = x$ and $y = -x^2 + 5x$ about the x-axis.

$$\pi \int_0^4 \left((-x^2 + 5x)^2 - x^2\right)\, dx = \pi \int_0^4 (x^4 - 10x^3 + 24x^2)\, dx = \frac{384}{5}\pi$$

5. Use the shell method to calculate the volume of the solid of revolution which is generated by the region bounded by the curves $y = e^{-x}$, $x = 0$, and $x = 2$, about the y-axis.

$$2\pi \int_0^2 xe^{-x}\, dx = 2\pi \left(-xe^{-x} - e^{-x}\right)\Big|_0^2 = 2\pi(-3e^{-2} + 1)$$

6. Determine the volume V of the solid generated by rotating the region bounded by the curves $y = 0$, $y = x^2 + x$ and $x = 2$ about the x-axis.

$$\pi \int_0^2 (x^2 + x)^2\, dx = \pi \int_0^2 (x^4 + 2x^3 + x^2)\, dx = \frac{256}{15}\pi$$

7. Find the area A of one petal of the three-petal rose $r(\theta) = \sin 3\theta$.

$$\int_0^{\pi/3} \frac{1}{2}\sin^2(3\theta)\, d\theta = \int_0^{\pi/3} \left(\frac{1}{4} - \frac{1}{4}\cos(6\theta)\right)\, d\theta = \left(\frac{1}{4}\theta - \frac{1}{24}\sin(6\theta)\right)\Big|_0^{\pi/3} = \frac{\pi}{12}$$

8. Find the area A of the region R which lies in the first quadrant and is bounded by the polar curves $r(\theta) = \theta$ and $r(\theta) = \sin\theta$.

$$\int_0^{\pi/2} \frac{1}{2}(\theta^2 - \sin^2\theta)\, d\theta = \int_0^{\pi/2} \frac{1}{2}\left(\theta^2 - \frac{1}{2} + \frac{1}{2}\cos(2\theta)\right)\, d\theta = \frac{1}{48}\pi^3 - \frac{1}{8}\pi$$

9. Find the area A and the centroid $(\overline{x}, \overline{y})$ of the infinite lamina that is bounded by the curves $x = 0$, $y = 0$, and $y = e^{-x}$.

$$A = \int_0^\infty e^{-x}\, dx = 1$$

$$\overline{x} = \frac{\int_0^\infty xe^{-x}\,dx}{\int_0^\infty e^{-x}\,dx} = \frac{1}{1} = 1$$

$$\overline{y} = \frac{\int_0^\infty \frac{1}{2}e^{-2x}\,dx}{\int_0^\infty e^{-x}\,dx} = \frac{1/4}{1} = \frac{1}{4}$$

10. A spherical tank of radius 10 ft is full of liquid of density ω lb/ft^3. Find the work W done by pumping the liquid to the top of the tank.

$$\omega\pi \int_{-10}^{0} (100 - y^2)(-y)\,dy = \omega\pi \int_{-10}^{0} (100y - y^3)(-y)\,dy = 24500000\,\omega\pi \;\; ft - lbs$$

11. A right circular tank of radius 4 ft and height 10 ft is half full of liquid whose density is ω lb/ft^3. Find the work W done by pumping the liquid to the top of the tank.

$$\int_0^5 16\,\omega\pi(10 - y)\,dy = 16\,\omega\pi\left(10y - \frac{1}{2}y^2\right)_0^5 = 600\,\omega\pi \;\; ft - lbs$$

12. Find the arc-length L of the given curve C:

(a) C is the cycloid $\vec{x}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, 1\rangle$, $0 \le t \le \pi$

$$\int_0^\pi \sqrt{t^2\cos^2 t + t^2\sin^2 t + 0}\,dt = \int_0^\pi t\,dt = \frac{1}{2}\pi^2$$

(b) C is the curve given by $\vec{x}(t) = \langle t - \sin t, 1 - \cos t, 2\rangle$, $0 \le t \le \pi$

$$\int_0^\pi \sqrt{2 - 2\cos t}\,dt = \int_0^\pi 2\sin(t/2)\,dt = 4$$

(c) C is the planar curve given by $y = 3x^{2/3}$, $0 \le x \le 8$.

$$\int_0^8 \sqrt{1 + \frac{4}{x^{2/3}}}\,dx = \int_0^8 \frac{\sqrt{x^{2/3} + 4}}{x^{1/3}}\,dx = (x^{2/3} + 4)^{3/2}\Big|_0^8 = 16\sqrt{2} - 8$$

(d) C is the planar curve given by $y = f(x)$, $-2 \le x \le 3$ where

$$f(x) = \begin{cases} x + 2 & x \le 0 \\ x^{3/2} + 2 & x \ge 0. \end{cases}$$

$$\int_{-2}^0 \sqrt{2}\,dx + \int_0^3 \frac{1}{2}\sqrt{4 + 9x}\,dx = 2\sqrt{2} + \frac{1}{27}((31)^{3/2} - (4)^{3/2})$$

13. Evaluate the improper integral if it converges. If it diverges, show reason for divergence.

(a) $\displaystyle\int_1^4 \frac{1}{(x-1)^2}\,dx = \left(-\frac{1}{x-1}\right)_{x=t\to 1}^{4} = \infty$ Diverges

(b) $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx = \frac{1}{2}(\ln x)^2\Big|_1^{x=t\to\infty} = \infty$ Diverges

(c) $\displaystyle\int_e^\infty \frac{1}{x\ln^2 x}\,dx = \left(-\frac{1}{\ln x}\right)_e^{x=t\to\infty} = 0-(-1) = 1$

(d) $\displaystyle\int_0^\infty \frac{x}{x^2+1}\,dx = \left(\frac{1}{2}\ln(x^2+1)\right)_0^{x=t\to\infty} = \infty$ Diverges

(e) $\displaystyle\int_0^\infty \frac{x}{(x^2+1)^2}\,dx = \frac{1}{2}\left(-\frac{1}{x^2+1}\right)_0^{x=t\to\infty} = \frac{1}{2}$

(f) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x|_0^{x=t\to 1} = \frac{\pi}{2}$

14. Determine whether the given infinite series is convergent or divergent. In case of convergence, specify if it is absolute or conditional. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{3n+2}{n^3+3}$ answer: Convergent because alternating and the terms are decreasing with a limit zero. Absolute convergence because $\displaystyle\sum_{n=1}^{\infty}\frac{3n+2}{n^3+3}$ compares and behaves like $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ which converges.

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n^2+19n}}$ answer: Convergent because alternating and the terms are decreasing with a limit zero. Conditional convergence because $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^2+19n}}$ compares and behaves like $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ which diverges.

(c) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n+1}}{n^2+2}$ answer: Convergent because alternating and the terms are decreasing with a limit zero. Absolute convergence because $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2+2}$ compares and behaves like $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which converges.

(d) $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+3}}{2n^2+4}$ answer: Divergent because $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+3}}{2n^2+4}$ compares and behaves like $\sum_{n=1}^{\infty} \frac{1}{2n}$ which diverges.

(e) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ answer: Convergent because alternating and the terms are decreasing with a limit zero. Absolute convergence because $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges to $e-1$ (or converges by ration test)

(f) $\sum_{n=1}^{\infty} \frac{2^n}{n^{24}}$ answer: Divergent because ratio tests yields a value of 2 which is greater then 1.

15. Find the sums of the given convergent series:

(a) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n!} = e^{-2} - 1$

(b) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

(c) $\sum_{n=2}^{\infty} \left(\frac{e}{\pi}\right) = \frac{e^2}{\pi(\pi - e)}$

16. Find the interval of convergence for the following power series:

(a) $\sum_{n=0}^{\infty} \frac{(x+1)^n}{(n+1)3^n}$ answer: $-4 \leq x < 2$.

(b) $\sum_{n=1}^{\infty} \frac{n(x+1)^n}{(n+1)!}$ answer: $-\infty < x < \infty$.

(c) $\sum_{n=0}^{\infty} \frac{(2x+1)^n}{n+1}$ answer: $-1 \leq x < 0$.

17. Find the Maclaurin expansion for $f(x) = xe^x + e^{2x}$.

$$x \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{x^{n+1} + 2^n x^n}{n!} \right)$$

18. Find the Taylor expansion for $f(x) = e^{3x}$, centered on $a = 2$.

$$\sum_{n=0}^{\infty} \frac{3^n e^6}{n!} (x-2)^n$$

19. Find the Maclaurin expansion, together with its radius of convergence, R, for the given function f:

(a) $f(x) = \dfrac{x}{1-x}$, $\qquad R = 1$

$$x \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+1}$$

(b) $f(x) = \dfrac{1}{(1+x)^2}$, $\qquad R = 1$

$$\sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1}$$

(c) $f(x) = \ln(1+x)$, $\qquad R = 1$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$$

20. Consider the three points $A(3,1,0)$, $B(2,0,-1)$, and $C(0,2,3)$.

(a) Find an equation of the plane determined by A, B, and C.
$\vec{AB} = \langle -1,-1,-1 \rangle$, $\vec{AC} = \langle -3,1,3 \rangle$, and $\vec{n} = \vec{AB} \times \vec{AC} = \langle -2,6,-4 \rangle$. Now take $\langle -2,6,-4 \rangle \cdot \langle x-3, y-1, z-0 \rangle = 0$

$$P: \qquad -2x + 6y - 4z = 0$$

(b) Find the area of the parallelogram with vertices A, B, and C.

$$A: \qquad ||\vec{n}|| = \sqrt{4+36+16} = \sqrt{56}$$

(c) Find the volume of the parallelepiped with vertices A, B, C, and $D(-1,0,1)$.

$\vec{AD} = \langle -4,-1,1\rangle$ Volume=2

(d) Find the projection of $\vec{AB}$ onto $\vec{AC}$.

$$\vec{P} = \frac{-1}{19}\langle -3,1,3\rangle$$

(e) Find the angle between $\vec{AB}$ and $\vec{AC}$.

$$\theta = \arccos\left(\frac{-1}{\sqrt{3}\sqrt{19}}\right)$$

21. Consider the line L which passes through the points $P(-1,4,3)$ and $Q(2,1,3)$. Find:

(a) An equation for the line L.
$\vec{PQ} = \vec{v} = \langle 3,-3,0\rangle$ L: $\vec{r}(t) = \langle -1,4,3\rangle + t\langle 3,-3,0\rangle$

(b) An equation for the line that is perpendicular to L and passes through the point $R(5,-2,2)$. (There are many answers)
$\vec{n} = \langle 1,1,1\rangle$ L_1: $\vec{s}(t) = \langle 5,-2,2\rangle + t\langle 1,1,1\rangle$

(c) An equation for the line that is parallel to L and passes through the point $R(5,-2,2)$. (There are many answers)
$\vec{p} = \langle 1,-1,0\rangle$ L_2: $\vec{s}(t) = \langle 5,-2,2\rangle + t\langle 1,-1,0\rangle$

22. Let T be the linear function given by:

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 8 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

and let P be the parallelogram with vertices at $(2,1)$, $(1,4)$, and $(4,8)$. Find the area of the parallelogram $T(P)$.
$\vec{PQ} = \langle 1-2, 4-1\rangle = \langle -1,3\rangle$ and $\vec{PR} = \langle 4-2, 8-1\rangle = \langle 2,7\rangle$

$$Area of P = \left| det\begin{pmatrix} -1 & 3 \\ 2 & 7 \end{pmatrix}\right| = 13$$

Area of $T(P) = |det(T)| \times Area(P) = |det(T)| \times 13 = 14(13) = 182$.

23. Let T be the linear function given by:

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and let L be the line described by

$$\vec{r}(t) = \langle 5, 0, -2\rangle + t\langle 4, 5, 2\rangle.$$

Find an equation for the line $T(L)$.

$T(L)$: $\quad \vec{s}(t) = \langle -10, 8, 17\rangle + t\langle -3, 15, 20\rangle$

24. Consider the line L_1 described by $\vec{r}(t) = \langle -2t + 1, 4t, -3t - 8\rangle$, and the line L_2 described by $\vec{s}(t) = \langle t + 4, -3t - 1, 2t - 6\rangle$.

(a) Show that L_1 and L_2 intersect, and find the point of intersection, $\vec{x_0}$.

$$\frac{x-1}{-2} = \frac{y}{4} = \frac{z+8}{-3}$$

$$\frac{x-4}{1} = \frac{y+1}{-3} = \frac{z+6}{2}$$

Solve for $x = 9$, $y = -16$ $z = 4$. So $\vec{x_0} = \langle 9, -16, 4\rangle$

(b) Using part (a), find an equation of the plane containing the lines L_1 and L_2.
$\vec{v_1} = \langle -2, 4, -3\rangle$ and $\vec{v_2} = \langle 1, -3, 2\rangle$. Therefore $\vec{n} = \vec{v_1} \times \vec{v_2} = \langle -1, 1, 2\rangle$

P : $\quad -x + y + 2z = -17$

25. Consider the curve C, given by $\vec{r}(t) = \langle t\sin t, t\cos t, t\rangle$.

(a) Find the velocity vector $\vec{v}$ and the acceleration vector $\vec{a}$, for each t.
$\vec{v}(t) = \langle \sin t + t\cos t, \cos t - t\sin t, 1\rangle$
$\vec{a}(t) = \langle 2\cos t - t\sin t, -2\sin t - t\cos t, 0\rangle$

(b) Find $v = ||\vec{v}||$ and $a = ||\vec{a}||$, for each t.
$||\vec{v}|| = \sqrt{t^2 + 1}$ and $||\vec{a}|| = \sqrt{t^2 + 4}$

(c) Find the tangential component a_τ and the normal component a_n of $\vec{a}$, for each t.
$\vec{T} = \dfrac{1}{\sqrt{t^2+1}}\langle \sin t + t\cos t, \cos t - t\sin t, 1\rangle$

$$a_\tau = \vec{a}\cdot\vec{T} = \frac{t}{\sqrt{t^2+1}}$$

$||\vec{a}||^2 = a_\tau^2 + a_n^2$

$$a_n = \frac{2+t^2}{\sqrt{t^2+1}}$$

(d) Use part (c) to find the curvature $\kappa = \kappa(t)$ for each t.
$\kappa(t) = \dfrac{1}{||\vec{v}(t)||^2}a_n$ so $\kappa(t) = \dfrac{t^2+2}{(t^2+1)^{3/2}}$.

26. Consider the curve C, given by $\vec{r}(t) = \langle t - \sin t, 1, 1-\cos t\rangle$, $0 \le t \le 2\pi$. Find:

(a) $\vec{v}$, $\vec{a}$, $v = ||\vec{v}||$, $a = ||\vec{a}||$, for $0 \le t \le 2\pi$.
$\vec{v}(t) = \langle 1-\cos t, 0, \sin t\rangle$, $\vec{a}(t) = \langle \sin t, 0, \cos t\rangle$
$||\vec{v}|| = \sqrt{2-2\cos t} = 2\sin(t/2)$, $||\vec{a}|| = 1$

(b) a_τ and a_n, for $0 \le t \le 2\pi$.
Note: $\langle 1-\cos t, 0, \sin t\rangle = \langle 2\sin^2(t/2), 0, 2\sin(t/2)\cos(t/2)\rangle$ by trig identities.
Therefore: $\vec{T} = \dfrac{1}{||\vec{v}||}\vec{v} = \langle \sin(t/2), 0, \cos(t/2)\rangle$

$$a_\tau = \vec{a}\cdot\vec{T} = \langle \sin t, 0, \cos t\rangle \cdot \langle \sin(t/2), 0, \cos(t/2)\rangle$$

$= \langle 2\sin(t/2)\cos(t/2), 0, \cos^2(t/2) - \sin^2(t/2)\rangle \cdot \langle \sin(t/2), 0, \cos(t/2)\rangle = \cos(t/2)$

$||\vec{a}||^2 = a_\tau^2 + a_n^2$ $a_n = \sqrt{1-\cos^2(t/2)} = \sin(t/2)$

(c) $\kappa = \kappa(t)$, for $0 \le t \le 2\pi$
$\kappa(t) = \dfrac{1}{||\vec{v}(t)||^2}a_n = \dfrac{1}{4\sin(t/2)}$

(d) $\vec{T}$, $\vec{N}$, and $\vec{B} = \vec{T}\times\vec{N}$
$\vec{T} = \langle \sin(t/2), 0, \cos(t/2)\rangle$
Let $\vec{P} = a_\tau\vec{T} = \cos(t/2)\langle \sin(t/2), 0, \cos(t/2)\rangle = \langle \frac{1}{2}\sin t, 0, \frac{1}{2}(1+\cos t)\rangle$
(note: $\vec{P}$ is the projection of $\vec{a}$ onto $\vec{T}$.)
Let $\vec{Q}$ be so that $\vec{a} = \vec{P} + \vec{Q}$. Then $\vec{Q} = \vec{a} - \vec{P} = \langle \frac{1}{2}\sin t, 0, \frac{1}{2}\cos t - \frac{1}{2}\rangle$

(note: $\vec{Q} = a_n \vec{N}$ where $\vec{N}$ is the unit normal)

Then $\vec{N} = \dfrac{1}{||\vec{Q}||}\vec{Q} = \dfrac{1}{\sin(t/2)}\langle \dfrac{1}{2}\sin t, 0, \dfrac{1}{2}\cos t - \dfrac{1}{2}\rangle = \langle \cos(t/2), 0, -\sin(t/2)\rangle$

$\vec{B} = \vec{T} \times \vec{N} = \langle 0, -1, 0\rangle$