High accuracy semi-implicit spectral deferred correction decoupling methods for a parabolic two domain problem

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Abstract

A numerical approach to estimating solutions to coupled systems of equations is partitioned time stepping methods, an alternative to monolithic solution methods, recently studied in the context of fluid-fluid and fluid-structure interaction problems. Few analytical results of stability and convergence are available, and typically such methods have been limited to first order accuracy in terms of discretization parameters. A brief overview of results is given in [11], and the computational evidence suggest that many proposed higher-order schemes are unstable, or their stability is yet to be proven analytically. This report considers two heat equations in $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ adjoined by an interface $I = \Omega_1 \cap \Omega_2 \subset \mathbb{R}$ - as a simplified model for the fluid-fluid or fluid-structure interactions. The family of semi-implicit spectral deferred correction (SISDC) methods for the partial differential equations is presented. The two-step SISDC method (one simpler method from this family) is then thoroughly discussed. The stability and the desired second-order accuracy are proven, and computations are provided verifying second-order time accuracy of the two-step method.

Keywords: semi-implicit, deferred correction, fluid-structure interaction, fluid-fluid interaction, ocean-atmosphere, implicit-explicit

1. Introduction

Numerical approximations are necessary for the numerous problems featuring multiple parameter regimes or coupled physical processes, where preexisting codes are considered the benchmark for solving the corresponding subproblems. Often such problems require coupling conditions to be satisfied on subdomain interfaces arising, for example, from conditions imposed to represent the interaction of physical processes. Solving the monolithic, coupled problem numerically via global discretizations may preclude usage of highly optimized black box subdomain solvers and limit accuracy and efficiency of computations. Iteratively solving the monolithic equations at each time step, decoupling is possible in the preconditioning and residual calculations and provide an attractive approach to some problems. Alternatively, partitioned time stepping methods provide a convenient decoupling strategy for large problems, allowing easy implementation of subdomain solvers. At each time step data is explicitly passed across the interface and the (decoupled) subdomain equations are then solved in parallel.

In this work, partitioned time stepping is applied to a simplified model of diffusion through two adjacent materials coupled across their shared interface I through a jump condition. This problem captures some of the time stepping difficulties in one candidate application, atmosphere-ocean interaction. The domain consists of two subdomains Ω_1 and Ω_2 coupled across an interface I (example in Figure 1 below). The problem is: *given* $\nu_i > 0, f_i : [0,T] \to H^1(\Omega_i), u_i(0) \in H^1(\Omega_i)$

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and $\kappa \in \mathbb{R}$, *find (for* $i = 1, 2$*)* $u_i : \Omega_i \times [0, T] \to \mathbb{R}$ *satisfying*

$$\begin{aligned} u_{i,t} - \nu_i \Delta u_i &= f_i, \quad \text{in } \Omega_i, &(1.1)\\ -\nu_i \nabla u_i \cdot \hat{n}_i &= \kappa(u_i - u_j), \quad \text{on } I,\ i, j = 1, 2\,,\, i \neq j\,, &(1.2)\\ u_i(x, 0) &= u_i^0(x), \quad \text{in } \Omega_i, &(1.3)\\ u_i &= g_i, \quad \text{on } \Gamma_i = \partial\Omega_i \setminus I. &(1.4) \end{aligned}$$

Let

$$X_i := \{v_i \in H^1(\Omega_i) : v_i = 0 \text{ on } \Gamma_i\}.$$

For $u_i \in X_i$ we denote $\mathbf{u} = (u_1, u_2)$ and $X := \{\mathbf{v} = (v_1, v_2) : v_i \in H^1(\Omega_i) : v_i = 0 \text{ on } \Gamma_i, i = 1, 2\}$. A natural subdomain variational formulation for (1.1)-(1.4), obtained by multiplying (1.1) by v_i, integrating and applying the divergence theorem, is to find (for $i, j = 1, 2, i \neq j$) $u_i : [0, T] \to X_i$ satisfying

$$(u_{i,t}, v_i)_{\Omega_i} + (\nu_i \nabla u_i, \nabla v_i)_{\Omega_i} + \int_I \kappa(u_i - u_j) v_i ds = (f_i, v_i)_{\Omega_i}, \text{ for all } v_i \in X_i. \quad (1.5)$$

The natural monolithic variational formulation for (1.1)-(1.4) is found by summing (1.5) over $i, j = 1, 2$ and $i \neq j$ and is to find $\mathbf{u} : [0, T] \to X$ satisfying

$$(\mathbf{u}_t, \mathbf{v}) + (\nu \nabla \mathbf{u}, \nabla \mathbf{v}) + \int_I \kappa [\mathbf{u}][\mathbf{v}] ds = (f, \mathbf{v}), \forall v \in X, \quad (1.6)$$

where $[\cdot]$ denotes the jump of the indicated quantity across the interface I , $(\cdot, \cdot)$ is the $L^2(\Omega_1 \cup \Omega_2)$ inner product and $\nu = \nu_i$ and $f = f_i$ in Ω_i.

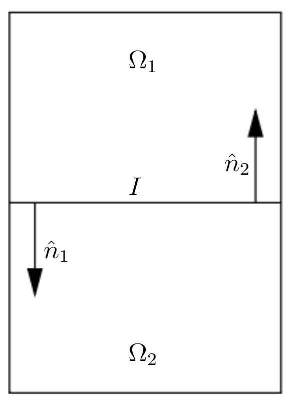

Figure 1: Example subdomains, coupled across an interface I.

Figure 1 illustrates the subdomains considered here, representative of commonly studied models in fluid-fluid and fluid-structure interaction, [7, 8, 11]. Comparing (1.6) and (1.5) we see that the monolithic problem (1.6) has a global energy that is exactly conserved, (in the appropriate sense), (set $\mathbf{v} = \mathbf{u}$ in (1.6)). The subdomain sub-problems (1.5) do not posses a subdomain energy which behaves similarly due to energy transfer back and forth across the interface I. It is possible for decoupling strategies to become unstable due to the input of non-physical energy as a numerical artifact.

Fluid-structure interaction problems, in particular blood flow models, are a typical application of partitioned methods. In these models the equations of elastic deformation of an arterial wall are coupled to equations of fluid flow through the vessel. Recently, it has been shown partitioned methods may be employed for this problem with the addition of a stabilization term on the fluid-structure interface. A defect correction step is implemented to recover optimal time accuracy, (see [8]).

Another application is atmosphere-ocean interaction, [3, 10, 7, 20]. This was a motivation for the work of Connors, Howell and Layton, [11], in developing partitioned schemes for (1.1)-(1.4), as a model capturing some of the technical difficulties of the coupled fluid-fluid problem. Two first order in time algorithms were analyzed, one an implicit-explicit (IMEX) approach where the interface term in the variational formulation is treated explicitly. The problem can also be discretized using the second and higher order IMEX schemes, [2]. However, stability results for the second order IMEX algorithm are not available. A brief overview of other approaches is given in [11], (see also [4, 9, 12]).

In this report, a second order in time, non-overlapping uncoupling method for (1.1)-(1.4) is presented: the two-step Semi-Implicit Spectral Deferred Correction (SISDC) method. At each step of the method the interface term in (1.5) is advanced in time to give one step black box decoupling of the subdomain problems in Ω_1 and Ω_2.

The main advantage of the deferred correction approach is that a simple low-order method can be employed, and the recovered solution is of high-order accuracy, due to a sequence of deferred correction equations. The general idea of defect correction and deferred correction methods for solving partial differential equations has been known for a long time, see the survey article [5]. Defect correction has proven computationally attractive in fluid applications. See Ervin and Lee [14] for the viscoelastic case and references therein for other defect correction work relevant to fluids.

The classical deferred correction approach could be seen, e.g., in [15]. However, in 2000 a modification of the classical deferred correction approach was introduced by Dutt, Greengard and Rokhlin, [13]. This allowed the construction of stable and high-order accurate *spectral deferred correction* methods. In [22] M.L. Minion discusses these spectral deferred correction (SDC) methods in application to an initial value ODE

$$\begin{aligned} \phi'(t) &= F(t, \phi(t)), \ t \in [a, b] \\ \phi(a) &= \phi_a. \end{aligned} \tag{1.7}$$

The solution is written in terms of the Picard integral equation; a polynomial is used to interpolate the subintegrand function and the obtained integral term is replaced by its quadrature approximation. In the case when the right hand side of the ODE can be decomposed into a sum of the stiff and non-stiff terms, a Semi-Implicit SDC method (SISDC) is introduces, which allows to treat the non-stiff terms explicitly and the stiff terms implicitly. These SISDC methods for solving ordinary differential equations are further discussed in [21].

The remainder of this work is organized as follows: in Section 2, notation and mathematical time-stepping algorithms are described: the family of the higher-order semi-implicit spectral deferred correction methods. Results regarding the stability of the two-step method are presented in Section 3. Convergence results are presented in Section 4, and computations are performed to investigate stability and accuracy of a two-step SISDC algorithm in section 5.

2. Method Description, Notation and Preliminaries

This section presents the numerical schemes for (1.1)-(1.4), and provides the necessary definitions and lemmas for the stability and convergence analysis. For $D \subset \Omega$, the Sobolev space $H^k(D) = W^{k,2}(D)$ is equipped with the usual norm $\|\cdot\|_{H^k(D)}$, and semi-norm $|\cdot|_{H^k(D)}$, for $1 \le k < \infty$, e.g. Adams [1]. The L^2 norm is denoted by $\|\cdot\|_D$. For functions $v(x,t)$ defined for almost every $t \in (0,T)$ on a function space $V(D)$, we define the norms $(1 \le p \le \infty)$

$$\|v\|_{L^\infty(0,T;V)} = \operatorname*{ess\,sup}_{0<t<T} \|v(\cdot,t)\|_V \quad \text{and} \quad \|v\|_{L^p(0,T;V)} = \left(\int_0^T \|v\|_V^p dt \right)^{1/p}.$$

The dual space of the Banach space V is denoted V'.

Let the domain $\Omega \subset \mathbb{R}^d$ (typically $d = 2, 3$) have convex, polygonal subdomains Ω_i for $i = 1, 2$ with $\partial\Omega_1 \cap \partial\Omega_2 = \Omega_1 \cap \Omega_2 = I$. Let Γ_i denote the portion of $\partial\Omega_i$ that is not on I, i.e. $\Gamma_i = \partial\Omega_i \setminus I$. For $i = 1, 2$, let $X_i = \{v \in H^1(\Omega_i) \mid v|_{\Gamma_i} = g_i\}$, let $(\cdot,\cdot)_{\Omega_i}$ denote the standard L^2 inner product on Ω_i, and let $(\cdot,\cdot)_{X_i}$ denote the standard H^1 inner product on Ω_i. Define $X = X_1 \times X_2$ and

$L^2(\Omega) = L^2(\Omega_1) \times L^2(\Omega_2)$ for $\mathbf{u}, \mathbf{v} \in X$ with $\mathbf{u} = [u_1, u_2]^T$ and $\mathbf{v} = [v_1, v_2]^T$, define the L^2 inner product

$$(\mathbf{u}, \mathbf{v}) = \sum_{i=1,2} \int_{\Omega_i} u_i v_i \, dx \,,$$

and H^1 inner product

$$(\mathbf{u}, \mathbf{v})_X = \sum_{i=1,2} \left(\int_{\Omega_i} u_i v_i \, dx + \int_{\Omega_i} \nabla u_i \cdot \nabla v_i \, dx \right) ,$$

and the induced norms $\|\mathbf{v}\| = (\mathbf{v}, \mathbf{v})^{1/2}$ and $\|\mathbf{v}\|_X = (\mathbf{v}, \mathbf{v})_X{}^{1/2}$, respectively. The case where $g_i = 0, i = 1, 2$ will be considered here, and can be easily extended to the case of nonhomogeneous Dirichlet conditions on $\partial\Omega_i \setminus I$.

Lemma 1. $(X, \|\cdot\|_X)$ *is a Hilbert space.*

PROOF. The choice of boundary conditions for X_1 and X_2 will ensure $X_i \subset H^1(\Omega_i)$, $i = 1, 2$ are closed subspaces. Hence by the definitions of $(\cdot, \cdot)_X$ and $\|\cdot\|_X$, $(X, \|\cdot\|_X)$ is a Hilbert space.

2.1. Discrete Formulation

Let $\mathcal{T}_i$ be a triangulation of Ω_i and $\mathcal{T}_h = \mathcal{T}_1 \cup \mathcal{T}_2$. Take $X_{i,h} \subset X_i$ to be conforming finite element spaces for $i = 1, 2$, and define $X_h = X_{1,h} \times X_{2,h} \subset X$. It follows that $X_h \subset X$ is a Hilbert space with corresponding inner product and induced norm. For $\mathbf{u} \in X$, define the operators $A, B : X \to (X)'$ via the Riesz Representation Theorem as

$$\big(A\mathbf{u}, \mathbf{v}\big) = \sum_{i=1,2} \nu_i \int_{\Omega_i} \nabla u_i : \nabla v_i \, dx, \quad \forall \mathbf{v} \in X \text{ and} \tag{2.1}$$

$$\big(B\mathbf{u}, \mathbf{v}\big) = \kappa \int_I [\mathbf{u}]\, [\mathbf{v}] \, ds, \;\; \forall \mathbf{v} \in X \,. \tag{2.2}$$

The discrete operators $A_h, B_h : X_h \to (X_h)' = X_h$ are defined analogously by restricting (2.1) and (2.2) to $\mathbf{v}_h \in X_h$. With this notation the coupled problem can be written

$$\frac{\partial \mathbf{u}}{\partial t} + A\mathbf{u} + B\mathbf{u} = \mathbf{f}, \; \mathbf{u}(x, 0) = \mathbf{u}_0. \tag{2.3}$$

For $t_k \in [0, T]$, $\mathbf{u}^k$ will denote the discrete approximation to $\mathbf{u}(t_k)$.

A standard partitioned time stepping approach for solving (2.3) is an IMEX scheme, see Connors, Howell, Layton, [11].

Algorithm 2.1 (First-order IMEX Scheme). *Let* $\Delta t > 0$, $\mathbf{f} \in L^2(\Omega)$. *For each* $M \in \mathbb{N}, M \leq \frac{T}{\Delta t}$, *given* $\mathbf{u}^n \in X_h$, $n = 0, 1, 2, \cdots, M-1$, *find* $\mathbf{u}^{n+1} \in X_h$ *satisfying*

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + A_h \mathbf{u}^{n+1} + B_h \mathbf{u}^n = \mathbf{f}(t^{n+1}), \tag{2.4}$$

or, in variational form,

$$\left(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t}, \mathbf{v} \right) + \big(A_h \mathbf{u}^{n+1}, \mathbf{v}\big) + (B_h \mathbf{u}^n, \mathbf{v}) = \big(\mathbf{f}(t^{n+1}), \mathbf{v}\big) \,, \;\; \forall \mathbf{v} \in X_h \,. \tag{2.5}$$

This scheme was extensively studied in [11] and was proven to be stable (provided $\Delta t \leq C \, \min\{\nu_1, \nu_2\} \, \kappa^{-2}$) and first order accurate. It will be shown in Section 3 that the SISDC method is also stable under these conditions (and higher order accurate).

The SISDC method constructs a sequence of approximations to the sought solution $\mathbf{u}$. The algorithm for the general family of SISDC applied to the model problem (2.3) is as follows.

Algorithm 2.2 (General SISDC). *Calculate* $\mathbf{u}_0, \mathbf{u}_1, ..., \mathbf{u}_m$ *- approximations to* $\mathbf{u}$ *via*

$$\frac{\mathbf{u}_0^{n+1} - \mathbf{u}_0^n}{\Delta t} + A_h \mathbf{u}_0^{n+1} + B_h \mathbf{u}_0^n = f^{n+1}, \tag{2.6a}$$

$$\frac{\mathbf{u}_{k+1}^{n+1} - \mathbf{u}_{k+1}^n}{\Delta t} + A_h \mathbf{r}_{k+1}^{n+1} + B_h \mathbf{r}_{k+1}^n = \frac{1}{\Delta t} I_n^{n+1}(\mathbf{u}_k), \text{ for } k = 0, 1, ..., m-1. \tag{2.6b}$$

Here $\mathbf{r}_{k+1}^i = \mathbf{u}_{k+1}^i - \mathbf{u}_k^i$, $k = 0, 1, ..., m-1$, $i = 0, 1, ..., N$.

$I_n^{n+1}(\mathbf{u}_k)$ is a numerical quadrature approximation to $\int_{t_n}^{t_{n+1}} F(\tau, \mathbf{u}_k(\tau)) d\tau$, where $F(t, \mathbf{u}) = f(t) - A_h \mathbf{u}(t) - B_h \mathbf{u}(t)$.

Remark 2.1. *Provided the integral terms* $I_n^{n+1}(\mathbf{u}_k)$ *are computed with the accuracy of order* $O((\Delta t)^{k+1})$*, after* k *iterations the above procedure will produce an approximate solution with global accuracy* $O((\Delta t)^{k+1})$*. If the points* $t_m \in [t_n, t_{n+1}]$ *are chosen to be Gaussian quadrature nodes, then the integral is being computed with a spectral integration rule, which is the reason for the name* ***spectral*** *deferred corrections.*

In this paper we will consider the two-step SISDC method and prove its stability and second order temporal accuracy. The method computes an approximation to the solution $\mathbf{u}$ of (2.3). At the first step the initial approximation $\mathbf{u}_0$ is computed via (2.6a). Thus, at the first step we use the Implicit Explicit scheme (Connors, Howell, Layton, [11]). For the second step we consider (2.6b) with $k = 0$. The second step approximation $\mathbf{u}_1$ satisfies

$$\frac{\mathbf{u}_1^{n+1} - \mathbf{u}_1^n}{\Delta t} + A_h \mathbf{u}_1^{n+1} + B_h \mathbf{u}_1^n = A_h \mathbf{u}_0^{n+1} + B_h \mathbf{u}_0^n + \frac{1}{\Delta t} I_n^{n+1}(\mathbf{u}_0). \tag{2.7}$$

It follows from Remark 2.1 that we need the numerical quadrature approximation of the integral term in (2.7) to be of the second order accuracy. We use Gaussian quadrature with one point - midpoint of the interval. Thus, the integral term in (2.7) is replaced by

$$\frac{F(t_n, \mathbf{u}_0) + F(t_{n+1}, \mathbf{u}_0)}{2} = \frac{f^{n+1} + f^n}{2} - \frac{A_h \mathbf{u}_0^{n+1} + A_h \mathbf{u}_0^n}{2} - \frac{B_h \mathbf{u}_0^{n+1} + B_h \mathbf{u}_0^n}{2}. \tag{2.8}$$

Therefore, in the case when the second order accuracy is sought and we only make two steps of the SISDC procedure, the second step equation could be written as:

$$\frac{\mathbf{u}_1^{n+1} - \mathbf{u}_1^n}{\Delta t} + A_h \mathbf{u}_1^{n+1} + B_h \mathbf{u}_1^n = \frac{f^{n+1} + f^n}{2} + \frac{\Delta t}{2} A_h (\frac{\mathbf{u}_0^{n+1} - \mathbf{u}_0^n}{\Delta t}) - \frac{\Delta t}{2} B_h (\frac{\mathbf{u}_0^{n+1} - \mathbf{u}_0^n}{\Delta t}). \tag{2.9}$$

The variational formulation of the two-step Semi-Implicit Spectral Deferred Correction method is:

Algorithm 2.3 (Two-Step SISDC Method). *Let* $\Delta t > 0$, $\mathbf{f} \in L^2(\Omega)$*. For each* $M \in \mathbb{N}, M \leq \frac{T}{\Delta t}$*, given* $\mathbf{u}_0^n, \mathbf{u}_1^n \in X_h$*,* $n = 0, 1, 2, \cdots, M-1$*, find* $\mathbf{u}_0^{n+1}, \mathbf{u}_1^{n+1} \in X_h$ *satisfying*

$$\left(\frac{\mathbf{u}_0^{n+1} - \mathbf{u}_0^n}{\Delta t}, \mathbf{v} \right) + \left(A_h \mathbf{u}_0^{n+1}, \mathbf{v} \right) + \left(B_h \mathbf{u}_0^n, \mathbf{v} \right) = \left(\mathbf{f}^{n+1}, \mathbf{v} \right), \ \forall \mathbf{v} \in X_h \tag{2.10}$$

$$\left(\frac{\mathbf{u}_1^{n+1} - \mathbf{u}_1^n}{\Delta t}, \mathbf{v} \right) + \left(A_h \mathbf{u}_1^{n+1}, \mathbf{v} \right) + \left(B_h \mathbf{u}_1^n, \mathbf{v} \right) = \left(\frac{\mathbf{f}^{n+1} + \mathbf{f}^n}{2}, \mathbf{v} \right) \tag{2.11}$$

$$+ \frac{\Delta t}{2} \left(A_h (\frac{\mathbf{u}_0^{n+1} - \mathbf{u}_0^n}{\Delta t}), \mathbf{v} \right) - \frac{\Delta t}{2} \left(B_h (\frac{\mathbf{u}_0^{n+1} - \mathbf{u}_0^n}{\Delta t}), \mathbf{v} \right), \ \forall \mathbf{v} \in X_h.$$

2.2. Analytical Tools

In this section results that will be utilized in the stability and convergence analysis are presented. It is necessary to work with norms induced by the operators A and B, and relate these norms back to $\|\cdot\|$ and $\|\cdot\|_X$. The next lemma serves to introduce useful norms for the numerical analysis and prove equivalence with the $\|\cdot\|_X$-norm. Originally presented in [11], the proof is included here to clarify subsequent arguments.

Lemma 2. *Let $\mathbf{v} = (v_1, v_2) \in X$ and $\alpha \geq 0$. The following is a norm on X:*

$$\|\mathbf{v}\|_{A+\alpha I} = \left\{ \sum_{i=1,2} \nu_i \int_{\Omega_i} |\nabla v_i|^2\, dx + \alpha \sum_{i=1,2} \int_{\Omega_i} |v_i|^2\, dx \right\}^{1/2} . \tag{2.12}$$

This norm is equivalent to $\|\cdot\|_X$. For $\alpha \geq C\,\kappa^2 \max\{\nu_1^{-1}, \nu_2^{-1}\}$,

$$\kappa \int_I [\mathbf{v}]^2\, ds \leq \sum_{i=1,2} \nu_i \int_{\Omega_i} |\nabla v_i|^2\, dx + \alpha \sum_{i=1,2} \int_{\Omega_i} |v_i|^2\, dx, \text{ and thus}$$

$$\|\mathbf{v}\|_{A+\alpha I-B} = \left\{ \sum_{i=1,2} \nu_i \int_{\Omega_i} |\nabla v_i|^2\, dx + \alpha \sum_{i=1,2} \int_{\Omega_i} |v_i|^2\, dx - \kappa \int_I |v_1 - v_2|^2\, ds \right\}^{1/2}$$

is a norm on X equivalent to $\|\cdot\|_X$.

Proof. The first assertion follows from noting the Poincaré–Friedrichs inequality holds on X_1 and X_2 under the boundary conditions, and thus that the norm is derived from an inner product on X. Then equivalence to the norm $\|\cdot\|_X$ is clear. It can also be shown that $\|\mathbf{v}\|_{A+\alpha I-B}$ is derived from an inner product by defining

$$(\mathbf{u}, \mathbf{v})_{A+\alpha I-B} = \sum_{i=1,2} \nu_i \int_{\Omega_i} \nabla u_i : \nabla v_i\, dx + \alpha \sum_{i=1,2} \int_{\Omega_i} u_i \cdot v_i\, dx - \kappa \int_I (u_1 - u_2)(v_1 - v_2)\, ds\, .$$

Linearity and symmetry are clear. It remains to prove definiteness and equivalence to $\|\cdot\|_A$. Note that

$$\begin{aligned}
\kappa \int_I |v_1 - v_2|^2\, ds &\leq \kappa \left\{ \|v_1\|^2_{L^2(I)} + 2\|v_1\|_{L^2(I)}\|v_2\|_{L^2(I)} + \|v_2\|^2_{L^2(I)} \right\} \\
&\leq 2\kappa \left\{ \|v_1\|^2_{L^2(I)} + \|v_2\|^2_{L^2(I)} \right\} \\
&= 2\kappa \left\{ \|v_1\|^2_{L^2(\partial\Omega_1)} + \|v_2\|^2_{L^2(\partial\Omega_2)} \right\} .
\end{aligned}$$

Application of the trace inequality [6] followed by Young's inequality yields

$$\begin{aligned}
\kappa \int_I |v_1 - v_2|^2\, ds &\leq C(\kappa, \Omega_1, \Omega_2) \left\{ \|v_1\|_{L^2(\Omega_1)} \|\nabla v_1\|_{L^2(\Omega_1)} + \|v_2\|_{L^2(\Omega_2)} \|\nabla v_2\|_{L^2(\Omega_2)} \right\} \\
&\leq C(\kappa, \Omega_1, \Omega_2) \left\{ \frac{1}{2\gamma_1} \|v_1\|^2_{L^2(\Omega_1)} + \frac{\gamma_1}{2} \|\nabla v_1\|^2_{L^2(\Omega_1)} + \frac{1}{2\gamma_2} \|v_2\|^2_{L^2(\Omega_2)} + \frac{\gamma_2}{2} \|\nabla v_2\|^2_{L^2(\Omega_2)} \right\} .
\end{aligned}$$

Choose $\gamma_i = \dfrac{\nu_i}{C(\kappa, \Omega_1, \Omega_2)}$ for $i = 1, 2$ and $\alpha = \dfrac{C(\kappa, \Omega_1, \Omega_2)^2}{2} \max\{\nu_1^{-1}, \nu_2^{-1}\}$. Then

$$\begin{aligned}
&\kappa \int_I |v_1 - v_2|^2\, ds \leq \alpha \|v_1\|^2_{L^2(\Omega_1)} + \frac{\nu_1}{2} \|\nabla v_1\|^2_{L^2(\Omega_1)} + \alpha \|v_2\|^2_{L^2(\Omega_2)} + \frac{\nu_2}{2} \|\nabla v_2\|^2_{L^2(\Omega_2)} \\
&\Rightarrow \frac{1}{2} \left\{ \nu_1 \|\nabla v_1\|^2_{L^2(\Omega_1)} + \nu_2 \|\nabla v_2\|^2_{L^2(\Omega_2)} \right\} \\
&\leq \sum_{i=1,2} \nu_i \int_{\Omega_i} |\nabla v_i|^2\, dx + \alpha \sum_{i=1,2} \int_{\Omega_i} |v_i|^2\, dx - \kappa \int_I |v_1 - v_2|^2\, ds \\
&\Rightarrow \frac{1}{2} \|\mathbf{v}\|^2_A \leq \|\mathbf{v}\|^2_{A+\alpha I-B} .
\end{aligned}$$

holds for this choice of $\alpha > 0$. This proves $(\mathbf{u}, \mathbf{u})_{A+\alpha I-B} = 0 \Leftrightarrow \mathbf{u} = 0$ for any $\mathbf{u} \in X$, and hence $\|\cdot\|_{A+\alpha I-B}$ is a norm on X. Last, to prove equivalence with $\|\cdot\|_A$, note that

$$\|\mathbf{v}\|_{A+\alpha I-B}^2 \leq \|\mathbf{v}\|_{A+\alpha I}^2 = \sum_{i=1,2} \left\{\alpha \|v_i\|_{L^2(\Omega_i)}^2 + \nu_i \|\nabla v_i\|_{L^2(\Omega_i)}^2\right\}$$
$$\leq \left\{1 + \alpha \max\left\{\frac{C_{PF}^2(\Omega_1)}{\nu_1}, \frac{C_{PF}^2(\Omega_2)}{\nu_2}\right\}\right\} \|\mathbf{v}\|_A^2 .$$

holds by applying the Poincaré - Friedrichs inequality.

The following discrete Gronwall lemma from [17] will also be utilized in the subsequent analysis.

Lemma 3. *Let k, M, and $a_\mu, b_\mu, c_\mu, \gamma_\mu$, for integers $\mu > 0$, be nonnegative numbers such that*

$$a_n + k\sum_{\mu=0}^{n} b_\mu \leq k\sum_{\mu=0}^{n} \gamma_\mu a_\mu + k\sum_{\mu=0}^{n} c_\mu + M \text{ for } n \geq 0. \tag{2.13}$$

Suppose that $k\gamma_\mu < 1$, for all μ, and set $\sigma_\mu \equiv (1 - k\gamma_\mu)^{-1}$. Then,

$$a_n + k\sum_{\mu=0}^{n} b_\mu \leq exp\left(k\sum_{\mu=0}^{n} \sigma_\mu \gamma_\mu\right)\left\{k\sum_{\mu=0}^{n} c_\mu + M\right\} \text{ for } n \geq 0. \tag{2.14}$$

Throughout the paper we use the following Modified H1 Projection.

Definition 1 (Modified H1 Projection). *The Modified H1 Projection operator $P\colon X \to X_h$, $P(\mathbf{u}) = \tilde{\mathbf{u}}$, satisfies*

$$((I + A + B)(\mathbf{u} - \tilde{\mathbf{u}}), \mathbf{v}^h) = 0, \tag{2.15}$$

for any $\mathbf{v}^h \in X_h$.

Proposition 4 (Stability of the Modified H1 Projection). *Let $\mathbf{u}$, $\tilde{\mathbf{u}}$ satisfy (2.15). Then there exists a constant $C = C(\kappa, \Omega_1, \Omega_2)$ such that*

$$\|\tilde{\mathbf{u}}\|^2 + \|\nabla \tilde{\mathbf{u}}\|^2 + (B\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) \leq C\|\mathbf{u}\|_X^2. \tag{2.16}$$

Proof. Take $\mathbf{v}^h = \tilde{\mathbf{u}} \in X_h$ in (2.15). Use Cauchy-Schwarz and Young's inequalities to obtain

$$\|\tilde{\mathbf{u}}\|^2 + \|\nabla \tilde{\mathbf{u}}\|^2 + (B\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) \leq \frac{1}{2}\|\tilde{\mathbf{u}}\|^2 + \frac{1}{2}\|\mathbf{u}\|^2$$
$$+\frac{1}{2}\|\nabla \tilde{\mathbf{u}}\|^2 + \frac{1}{2}\|\nabla \mathbf{u}\|^2 + (B\mathbf{u}, \tilde{\mathbf{u}}).$$

Using the trace inequality as in the proof of Lemma 2 one can show that

$$(B\mathbf{u}, \tilde{\mathbf{u}}) \leq \frac{1}{4}\|\tilde{\mathbf{u}}\|^2 + \frac{1}{4}\|\nabla \tilde{\mathbf{u}}\|^2 + C(\kappa, \Omega_1, \Omega_2)\|\mathbf{u}\|_X^2,$$

which concludes the proof.

In the error analysis we shall use the error estimate of the Modified H1 Projection (2.15).

Proposition 5 (Error estimate for the Modified H1 Projection). *The error in the Modified H1 Projection satisfies*

$$\|\mathbf{u} - \tilde{\mathbf{u}}\|^2 + \|\nabla(\mathbf{u} - \tilde{\mathbf{u}})\|^2 + (B(\mathbf{u} - \tilde{\mathbf{u}}), \mathbf{u} - \tilde{\mathbf{u}}) \leq C \inf_{\mathbf{v}^h \in X_h} \|(\mathbf{u} - \mathbf{v}^h)\|_X^2, \tag{2.17}$$

where $C = C(\kappa, \Omega_1, \Omega_2)$.

Proof. Decompose the projection error $\mathbf{u}-\tilde{\mathbf{u}} = \mathbf{u}-I(\mathbf{u})+(I(\mathbf{u})-\tilde{\mathbf{u}}) = \boldsymbol{\eta}+\boldsymbol{\phi}$, where $\boldsymbol{\eta} = \mathbf{u}-I(\mathbf{u})$, $\boldsymbol{\phi} = I(\mathbf{u}) - \tilde{\mathbf{u}}$, and $I(\mathbf{u})$ approximates $\mathbf{u}$ in X_h. Take $\mathbf{v}^h = \boldsymbol{\phi} \in X_h$ in (2.15). This gives

$$\|\boldsymbol{\phi}\|^2 + \|\nabla\boldsymbol{\phi}\|^2 + (B\boldsymbol{\phi},\boldsymbol{\phi}) = -(\boldsymbol{\eta},\boldsymbol{\phi}) - (\nabla\boldsymbol{\eta},\nabla\boldsymbol{\phi}) - (B\boldsymbol{\eta},\boldsymbol{\phi}). \tag{2.18}$$

Using the trace inequality as in the proof of Lemma 2 and applying the Cauchy-Schwarz and Young's inequalities leads to

$$\|\boldsymbol{\phi}\|^2 + \|\nabla\boldsymbol{\phi}\|^2 + (B\boldsymbol{\phi},\boldsymbol{\phi}) \le C\|\boldsymbol{\eta}\|_X^2. \tag{2.19}$$

Since $I(\mathbf{u})$ is an approximation of $\mathbf{u}$ in X_h, we can take infimum over X_h. The proof is concluded by applying the triangle inequality.

3. Stability

Stability of the IMEX scheme (Algorithm 2.1) was established in [11], see the lemma below.

Lemma 6. *(IMEX Stability) Let $\mathbf{u}_0^{n+1} \in X^h$ satisfy (2.10) for each $n \in \left\{0,1,2,\cdots,\frac{T}{\Delta t}-1\right\}$, and $0<\Delta t<(2\alpha+1)^{-1}$. Then $\exists C_1, C_2 > 0$ independent of h, Δt such that $\mathbf{u}_0^{n+1}$ satisfies:*

$$\left\|\mathbf{u}_0^{n+1}\right\|^2 + \Delta t\sum_{k=0}^{n+1}\|\mathbf{u}_0^k\|_X^2 \le C_1(\alpha)e^{C_2(\alpha)T}\left\{\|\mathbf{u}_0^0\|^2 + \Delta t\|\mathbf{u}_0^0\|_X^2 + \Delta t\sum_{k=0}^{n}\|\mathbf{f}(t^{k+1})\|^2\right\}.$$

Hence, the initial approximation $\mathbf{u}_0$ which satisfies (2.10) is stable. We conclude the proof of stability of the SISDC approximations by considering the second step approximation $\mathbf{u}_1$ satisfying (2.11).

Theorem 7 (Stability of SISDC). *Let $\mathbf{u}_1^{n+1} \in X^h$ satisfy (2.11) for each $n \in \left\{0,1,2,\cdots,\frac{T}{\Delta t}-1\right\}$, and $0<\Delta t<(2\alpha+1)^{-1}$. Then $\exists C_1, C_2 > 0$ independent of h, Δt such that $\mathbf{u}_1^{n+1}$ satisfies:*

$$\left\|\mathbf{u}_1^{n+1}\right\|^2 + \Delta t\sum_{k=0}^{n+1}\|\mathbf{u}_1^k\|_X^2 \le C_1(\alpha)e^{C_2(\alpha)T}\left\{\|\mathbf{u}_1^0\|^2 + \Delta t\|\mathbf{u}_1^0\|_X^2 + \Delta t\sum_{k=0}^{n}\|\mathbf{f}(t^{k+1})\|^2\right\}.$$

Proof. Choose $\mathbf{v} = \mathbf{u}_1^{n+1}$ in (2.11). Then it follows:

$$\begin{aligned}\left(\frac{\mathbf{u}_1^{n+1}-\mathbf{u}_1^n}{\Delta t},\mathbf{u}_1^{n+1}\right) + \left(A_h\mathbf{u}_1^{n+1},\mathbf{u}_1^{n+1}\right) + \left(B_h\mathbf{u}_1^{n},\mathbf{u}_1^{n+1}\right) &= \left(\frac{\mathbf{f}^{n+1}+\mathbf{f}^n}{2},\mathbf{u}_1^{n+1}\right)\\ &+\frac{\Delta t}{2}\left(A_h(\frac{\mathbf{u}_0^{n+1}-\mathbf{u}_0^n}{\Delta t}),\mathbf{u}_1^{n+1}\right) - \frac{\Delta t}{2}\left(B_h(\frac{\mathbf{u}_0^{n+1}-\mathbf{u}_0^n}{\Delta t}),\mathbf{u}_1^{n+1}\right).\end{aligned}$$

Add $\alpha(\mathbf{u}_1^{n+1},\mathbf{u}_1^{n+1})$ to both sides and apply (2.12). Then apply Young's inequality using the fact that

$$\left(B_h\mathbf{u}_1^n,\mathbf{u}_1^{n+1}\right) \ge -\frac{1}{2}\left(B_h\mathbf{u}_1^{n+1},\mathbf{u}_1^{n+1}\right) - \frac{1}{2}\left(B_h\mathbf{u}_1^n,\mathbf{u}_1^n\right).$$

Then split the term

$$\|\mathbf{u}_1^{n+1}\|_{A+\alpha I}^2 = \frac{1}{2}\|\mathbf{u}_1^{n+1}\|_{A+\alpha I}^2 + \frac{1}{2}\left(\|\mathbf{u}_1^{n+1}\|_{A+\alpha I}^2 - \|\mathbf{u}_1^n\|_{A+\alpha I}^2\right) + \frac{1}{2}\|\mathbf{u}_1^n\|_{A+\alpha I}^2.$$

These results together with Lemma 2 imply the new estimate

$$\begin{aligned}\frac{1}{2\Delta t}\left(\|\mathbf{u}_1^{n+1}\|^2 - \|\mathbf{u}_1^n\|^2\right) + \frac{1}{2}\|\mathbf{u}_1^{n+1}\|_{A+\alpha I-B}^2 + \frac{1}{2}\left(\|\mathbf{u}_1^{n+1}\|_{A+\alpha I}^2 - \|\mathbf{u}_1^n\|_{A+\alpha I}^2\right) + \frac{1}{2}\|\mathbf{u}_1^n\|_{A+\alpha I-B}^2\\ \le \left(\frac{\mathbf{f}^{n+1}+\mathbf{f}^n}{2},\mathbf{u}_1^{n+1}\right) + \alpha\|\mathbf{u}_1^{n+1}\|^2 + \frac{1}{4}\|\mathbf{u}_1^{n+1}\|_{A+\alpha I-B}^2 + C(\|\mathbf{u}_0^{n+1}\|_X^2 + \|\mathbf{u}_0^n\|_X^2).\end{aligned}$$

Rearranging terms,

$$\|\mathbf{u}_1^{n+1}\|^2 + \Delta t\|\mathbf{u}_1^{n+1}\|^2_{A+\alpha I} + \Delta t\sum_{k=0}^{n}\left\{\|\mathbf{u}_1^{k+1}\|^2_{A+\alpha I-B} + \|\mathbf{u}_1^k\|^2_{A+\alpha I-B}\right\}$$
$$\leq \|\mathbf{u}_1^0\|^2 + \Delta t\|\mathbf{u}_1^0\|^2_{A+\alpha I} + \Delta t\sum_{k=0}^{n}(\|\mathbf{u}_0^{k+1}\|^2_X + \|\frac{\mathbf{f}^{k+1}+\mathbf{f}^k}{2}\|^2) + \Delta t(2\alpha+1)\sum_{k=0}^{n}\|\mathbf{u}_1^{k+1}\|^2.$$

Take $\gamma_n \equiv 2\alpha + 1$ in Lemma 3. Choose $C_2(\alpha) = 2(2\alpha+1)(1-\Delta t(2\alpha+1)^{-1})$. Applying Lemma 6 and Lemma 2 concludes the proof.

4. Convergence analysis

The convergence analysis for Algorithm 2.1 was performed in [11], see the theorem below.

Theorem 8. *(Convergence of the IMEX scheme) Let $\mathbf{u}(t;x) \in X$ for all $t \in (0,T)$ solve (1.1)–(1.4), such that $\mathbf{u}_t \in L^2(0,T;X)$ and $\mathbf{u}_{tt} \in L^2(0,T;L^2(\Omega))$. Then $\exists C_1, C_2 > 0$ independent of h, Δt such that for any $n \in \{0,1,2,\cdots,M-1 = \frac{T}{\Delta t}-1\}$ and $0 < \Delta t < (2+2\alpha)^{-1}$, the solution $\mathbf{u}_0^{n+1} \in X_h$ of (2.10) satisfies:*

$$\|\mathbf{u}(t^{n+1}) - \mathbf{u}_0^{n+1}\|^2 + \Delta t\|\mathbf{u}(t^{n+1}) - \mathbf{u}_0^{n+1}\|^2_X + \frac{3\Delta t}{4}\sum_{k=0}^{n}\|\mathbf{u}(t^{k+1}) - \mathbf{u}_0^{k+1}\|^2_X$$
$$\leq C_1(\alpha)e^{C_2(\alpha)T}\Big\{\|\mathbf{u}(0) - \mathbf{u}_0^0\|^2 + \Delta t\|\mathbf{u}(0) - \mathbf{u}_0^0\|^2_X + \Delta t^2\|\mathbf{u}_t\|^2_{L^2(0,T;X)}$$
$$+ \Delta t^2\|\mathbf{u}_{tt}\|^2_{L^2(0,T;L^2(\Omega))}$$
$$+ \inf_{\mathbf{v}^0\in X_h}\left\{\|\mathbf{u}(0) - \mathbf{v}^0\|^2 + \Delta t\|\mathbf{u}(0) - \mathbf{v}^0\|^2_X\right\} + \inf_{\mathbf{v}\in X_h}\|(\mathbf{u}(0) - \mathbf{v})_t\|^2$$
$$+ T\max_{k=1,2,\cdots,n+1}\inf_{\mathbf{v}^k\in X_h}\|\mathbf{u}(t^k) - \mathbf{v}^k\|^2_X\Big\}.$$

Corollary 4.1. *Let $X_h \subset X$ be a finite element space corresponding to continuous piece-wise polynomials of degree k. If $\mathbf{u}(\cdot,t)$ is a solution of (1.1)–(1.4) satisfying the assumptions of Theorem 8, and $\mathbf{u}_0^0$ approximates $\mathbf{u}(\cdot,0)$ such that*

$$\|\mathbf{u}(\cdot,0) - \mathbf{u}_0^0\| = O(h^q),$$

then the approximation (2.10) converges at the rate $O(\Delta t + h^q)$ in the norm

$$\left\{\Delta t\sum_{k=0}^{M}\|\mathbf{u}(t^k) - \mathbf{u}_0^k\|^2_X\right\}^{1/2}.$$

The rest of this section will be devoted to deriving a bound on the error in the second step approximation $\mathbf{u} - \mathbf{u}_1$. Let $\mathbf{e}_j^i = \mathbf{u}(t_i) - \mathbf{u}_j^i$, $\forall i = 0,1,...,N$, $j = 1,2$. The bounds on $\mathbf{e}_0^i$ have been derived in Theorem 8. We now need the bounds on $\frac{\mathbf{e}_0^{i+1}-\mathbf{e}_0^i}{\Delta t}$.

Theorem 9 (IMEX time derivative). *Let $\mathbf{u}(t;x) \in X$ for all $t \in (0,T)$ solve (1.1)–(1.4), such that $\mathbf{u}_t \in L^2(0,T;X)$, $\mathbf{u}_{tt} \in L^2(0,T;L^2(\Omega))$ and $\mathbf{u}_{ttt} \in L^2(0,T;L^2(\Omega))$. Then $\exists C > 0$ independent of h, Δt such that for any $n \in \{0,1,2,\cdots,M-1 = \frac{T}{\Delta t}-1\}$ and $0 < \Delta t < (2+2\alpha)^{-1}$, the discrete time derivative of the error $\frac{\mathbf{e}_0^{i+1}-\mathbf{e}_0^i}{\Delta t}$ satisfies:*

$$\|\frac{\mathbf{e}_0^{n+1}-\mathbf{e}_0^n}{\Delta t}\|^2 + \Delta t\sum_{i=0}^{n}\|\frac{\mathbf{e}_0^{i+1}-\mathbf{e}_0^i}{\Delta t}\|^2_X$$
$$\leq C[\|\mathbf{e}_0^{n+1}\|^2 + \Delta t\sum_{i=0}^{n}\|\mathbf{e}_0^{i+1}\|^2_X].$$

PROOF. Restricting test functions to X_h, subtract (2.10) from (2.3) to get the error equation. Let $\mathbf{e}_0^n = \boldsymbol{\eta}^n + \boldsymbol{\phi}^n$, where $\boldsymbol{\eta}^n = \mathbf{u}(t_n) - \mathbf{v}^h$, $\boldsymbol{\phi}^n = \mathbf{v}^h - \mathbf{u}_0^n$, for some $\mathbf{v}^h \in X_h$. Then for $\forall n \geq 0$:

$$\begin{aligned}(\frac{\boldsymbol{\phi}^{n+1} - \boldsymbol{\phi}^n}{\Delta t}, \mathbf{v}) + \big(A\boldsymbol{\phi}^{n+1}, \mathbf{v}\big) + \big(B(\mathbf{u}(t^{n+1}) - \mathbf{u}_0^n), \mathbf{v}\big) \\ = -\left(\frac{\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n}{\Delta t}, \mathbf{v}\right) - \big(\mathbf{r}^{n+1}, \mathbf{v}\big) - \big(A\boldsymbol{\eta}^{n+1}, \mathbf{v}\big)\,,\ \forall \mathbf{v} \in X_h,\end{aligned} \tag{4.1}$$

where $\mathbf{r}^{n+1} = \mathbf{u}_t(t^{n+1}) - \frac{\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)}{\Delta t}$.

In order to treat the B-term, add and subtract $B\mathbf{u}(t^k)$; it follows that

$$B\mathbf{u}(t^{n+1}) - B\mathbf{u}_0^n = B(\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)) + B\boldsymbol{\eta}^n + B\boldsymbol{\phi}^n.$$

Consider (4.1) at the current time level $n+1$ and the previous time level n. Subtract the latter from the current time level, making the same choice $\mathbf{v} = \frac{\boldsymbol{\phi}^{n+1} - \boldsymbol{\phi}^n}{\Delta t}$ in both equations. Denoting $s^{n+1} = \frac{\boldsymbol{\phi}^{n+1} - \boldsymbol{\phi}^n}{\Delta t}$ for $\forall n \geq 0$, we obtain, after dividing by Δt:

$$\begin{aligned}(\frac{s^{n+1} - s^n}{\Delta t}, s^{n+1}) + (As^{n+1}, s^{n+1}) + \Delta t(B(\frac{\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}}{(\Delta t)^2}), s^{n+1}) \\ +(B(\frac{\boldsymbol{\eta}^n - \boldsymbol{\eta}^{n-1}}{\Delta t}), s^{n+1}) + (Bs^n, s^{n+1}) = -(\frac{\boldsymbol{\eta}^{n+1} - 2\boldsymbol{\eta}^n + \boldsymbol{\eta}^{n-1}}{(\Delta t)^2}, s^{n+1}) \\ -(A(\frac{\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n}{\Delta t}), s^{n+1}) - (\frac{\mathbf{r}^{n+1} - \mathbf{r}^n}{\Delta t}, s^{n+1}).\end{aligned} \tag{4.2}$$

Replacing $\boldsymbol{\phi}$ by s in the IMEX error equation (4.1) results exactly in (4.2), but the regularity assumptions are now needed for $\mathbf{u}_{ttt}$ instead of $\mathbf{u}_{tt}$. Hence, the result analogous to Theorem 8 can be obtained by an argument similar to the proof of Theorem 8. However, the summation is now possible only up to $s = 1$, leaving two extra terms in the right hand side:

$$\begin{aligned}\|\frac{\mathbf{e}_0^{n+1} - \mathbf{e}_0^n}{\Delta t}\|^2 + \Delta t \sum_{i=0}^{n} \|\frac{\mathbf{e}_0^{i+1} - \mathbf{e}_0^i}{\Delta t}\|_X^2 \leq C[\|\frac{\boldsymbol{\phi}^1 - \boldsymbol{\phi}^0}{\Delta t}\|^2 \\ +\Delta t\|\nabla(\frac{\boldsymbol{\phi}^1 - \boldsymbol{\phi}^0}{\Delta t})\|^2 + (\Delta t)^2\|\mathbf{u}_{ttt}\|^2_{L^2(0,T;L^2(\Omega))} + \|\mathbf{e}_0^{n+1}\|^2 + \Delta t \sum_{i=0}^{n} \|\mathbf{e}_0^{i+1}\|_X^2].\end{aligned} \tag{4.3}$$

Consider (4.1) at the initial time level $n = 0$. Take $\mathbf{v} = \frac{\boldsymbol{\phi}^1 - \boldsymbol{\phi}^0}{\Delta t} \in X_h$. Choose the initial time level approximation $\mathbf{u}_0^0 \in X_h$ to be the Modified H1 Projection of $\mathbf{u}(x,0) \in X$: $P(\mathbf{u}(x,0)) = \mathbf{u}_0^0$. It follows from (2.15) that

$$(A(\boldsymbol{\eta}^0 + \boldsymbol{\phi}^0) + B(\boldsymbol{\eta}^0 + \boldsymbol{\phi}^0), \frac{\boldsymbol{\phi}^1 - \boldsymbol{\phi}^0}{\Delta t}) = -(\boldsymbol{\eta}^0 + \boldsymbol{\phi}^0, \frac{\boldsymbol{\phi}^1 - \boldsymbol{\phi}^0}{\Delta t}).$$

Thus, we obtain from (4.1) that

$$\begin{aligned}\|\frac{\boldsymbol{\phi}^1 - \boldsymbol{\phi}^0}{\Delta t}\|^2 + \Delta t\|\nabla(\frac{\boldsymbol{\phi}^1 - \boldsymbol{\phi}^0}{\Delta t})\|^2 \\ +\Delta t(B(\frac{\mathbf{u}^1 - \mathbf{u}^0}{\Delta t}), \frac{\boldsymbol{\phi}^1 - \boldsymbol{\phi}^0}{\Delta t}) + \Delta t(A(\frac{\boldsymbol{\eta}^1 - \boldsymbol{\eta}^0}{\Delta t}), \frac{\boldsymbol{\phi}^1 - \boldsymbol{\phi}^0}{\Delta t}) = 0.\end{aligned} \tag{4.4}$$

Therefore, using Cauchy-Schwarz and Young's inequalities, we get

$$\|\frac{\boldsymbol{\phi}^1 - \boldsymbol{\phi}^0}{\Delta t}\|^2 + \Delta t\|\nabla(\frac{\boldsymbol{\phi}^1 - \boldsymbol{\phi}^0}{\Delta t})\|^2 \leq C(\Delta t)^2\|\mathbf{u}_t\|_X^2.$$

This result combined with (4.3) concludes the proof.

Finally, we have derived all the intermediate results necessary for the proof of the main theorem of this section.

Theorem 10 (SISDC error estimate). *Let the conditions of the Theorem 9 be satisfied. Let also* $\mathbf{u}_t \in L^2(0,T;X)$ *and* $\mathbf{u}_{ttt} \in L^2(0,T;L^2(\Omega))$*. Then* $\exists C > 0$ *independent of* h, Δt *such that for any* $n \in \{0,1,2,\cdots,M-1 = \frac{T}{\Delta t} - 1\}$ *and* $0 < \Delta t < (2+2\alpha)^{-1}$*, the second step approximation error* $\mathbf{e}_1^{i+1}$ *satisfies:*

$$\|\mathbf{u}(t^{n+1}) - \mathbf{u}_1^{n+1}\|^2 + \Delta t\|\mathbf{u}(t^{n+1}) - \mathbf{u}_1^{n+1}\|_X^2 + \Delta t\sum_{k=0}^{n}\|\mathbf{u}(t^{k+1}) - \mathbf{u}_1^{k+1}\|_X^2$$
$$\leq C\Big\{\|\mathbf{u}(0) - \mathbf{u}_1^0\|^2 + \Delta t\|\mathbf{u}(0) - \mathbf{u}_1^0\|_X^2 + \Delta t^4\|\mathbf{u}_t\|_{L^2(0,T;X)}^2$$
$$+ \Delta t^4\|\mathbf{u}_{tt}\|_{L^2(0,T;L^2(\Omega))}^2 + \Delta t^4\|\mathbf{u}_{ttt}\|_{L^2(0,T;L^2(\Omega))}^2$$
$$+ \inf_{\mathbf{v}^0\in X_h}\{\|\mathbf{u}(0) - \mathbf{v}^0\|^2 + \Delta t\|\mathbf{u}(0) - \mathbf{v}^0\|_X^2\} + \inf_{\mathbf{v}\in X_h}\|(\mathbf{u}(0) - \mathbf{v})_t\|^2$$
$$+ T\max_{k=1,2,\cdots,n+1}\inf_{\mathbf{v}^k\in X_h}\|\mathbf{u}(t^k) - \mathbf{v}^k\|_X^2\Big\}.$$

Corollary 4.2. *Let* $X_h \subset X$ *be a finite element space corresponding to continuous piece-wise polynomials of degree* k*. If* $\mathbf{u}(\cdot,t)$ *is a solution of (1.1)–(1.4) satisfying the assumptions of Theorem 8, and* $\mathbf{u}_0^0, \mathbf{u}_1^0$ *approximates* $\mathbf{u}(\cdot,0)$ *such that*

$$\|\mathbf{u}(\cdot,0) - \mathbf{u}_0^0\| = O(h^q),$$
$$\|\mathbf{u}(\cdot,0) - \mathbf{u}_1^0\| = O(h^q),$$

then the approximation (2.11) converges at the rate $O((\Delta t)^2 + h^q + h^k)$ *in the norm*

$$\left\{\Delta t\sum_{k=0}^{M}\|\mathbf{u}(t^k) - \mathbf{u}_1^k\|_X^2\right\}^{1/2}.$$

Proof. The equation for the true solution (2.3) could be written as

$$(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t}, \mathbf{v}) + (A\mathbf{u}^{n+1}, \mathbf{v}) + (B\mathbf{u}^n, \mathbf{v}) = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} F(\tau, \mathbf{u})d\tau \tag{4.5}$$
$$+(A\mathbf{u}^{n+1}, \mathbf{v}) + (B\mathbf{u}^n, \mathbf{v}), \forall \mathbf{v} \in X.$$

The Gaussian quadrature rule with one point (midpoint of $[t_n, t_{n+1}]$) gives

$$\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} F(\tau, \mathbf{u})d\tau = \frac{F(t_n, \mathbf{u}) + F(t_{n+1}, \mathbf{u})}{2} + C(\Delta t)^2F_{tt}(\xi, \mathbf{u}) \tag{4.6}$$
$$= \frac{\mathbf{f}^{n+1} + \mathbf{f}^n}{2} - A(\frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2}) - B(\frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2}) + C(\Delta t)^2F_{tt}(\xi, \mathbf{u}),$$

for some $\xi \in [t_n, t_{n+1}]$. Hence, it follows from (4.5)-(4.6) that the equation for the true solution $\mathbf{u}$ can be written as:

$$(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t}, \mathbf{v}) + (A\mathbf{u}^{n+1}, \mathbf{v}) + (B\mathbf{u}^n, \mathbf{v}) = (\frac{\mathbf{f}^{n+1} + \mathbf{f}^n}{2}, \mathbf{v}) \tag{4.7}$$
$$+\frac{\Delta t}{2}(A(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t}), \mathbf{v}) - \frac{\Delta t}{2}(B(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t}), \mathbf{v}) + C(\Delta t)^2(F_{tt}(\xi, \mathbf{u}), \mathbf{v}), \forall \mathbf{v} \in X.$$

Subtracting (2.11) from (4.7) gives

$$(\frac{\mathbf{e}_1^{n+1} - \mathbf{e}_1^n}{\Delta t}, \mathbf{v}) + (A\mathbf{e}_1^{n+1}, \mathbf{v}) + (B\mathbf{e}_1^n, \mathbf{v}) = \frac{\Delta t}{2}(A(\frac{\mathbf{e}_0^{n+1} - \mathbf{e}_0^n}{\Delta t}), \mathbf{v}) \tag{4.8}$$
$$-\frac{\Delta t}{2}(B(\frac{\mathbf{e}_0^{n+1} - \mathbf{e}_0^n}{\Delta t}), \mathbf{v}) + C(\Delta t)^2(F_{tt}(\xi, \mathbf{u}), \mathbf{v}), \forall \mathbf{v} \in X_h.$$

Take $\mathbf{v} = \mathbf{e}_1^{n+1}$. The Cauchy-Schwarz and Young inequalities, together with the results of Lemma 2 and Theorem 9 complete the proof.

5. Computational Testing

The convergence properties of the two-step SISDC method (Algorithm 2.3) are investigated here in the case of a test problem previously discussed in [11] using the first-order in time IMEX method. Emphasis is placed on understanding time accuracy and errors related to the interface. Each iteration of Algorithm 2.3 performs a first order in time IMEX half-step followed by a correction half-step to obtain second order in time accuracy. Thus the output of the algorithm is compared between the uncorrected and corrected steps.

Assume $\Omega_1 = [0,1]\times[0,1]$ and $\Omega_2 = [0,1]\times[-1,0]$, so I is the portion of the x-axis from 0 to 1. Then $\mathbf{n}_1 = [0,\ -1]^T$ and $\mathbf{n}_2 = [0,\ 1]^T$. For ν_1, ν_2, and κ all arbitrary positive constants, the right hand side function $\mathbf{f}$ from (1.1) is calculated by differentiating

$$\begin{aligned} u_1(t,x,y) &= x(1-x)(1-y)e^{-t} \\ u_2(t,x,y) &= x(1-x)(1+\frac{\nu_1}{\kappa} - \frac{\nu_1}{\nu_2}y - (1+\frac{\nu_1}{\nu_2}+\frac{\nu_1}{\kappa})y^2)e^{-t}\,. \end{aligned}$$

This choice of $\mathbf{u}$ satisfies the interface conditions (1.2) and the boundary conditions (1.4) with $g_1 = g_2 = 0$. Choosing κ to be no larger than ν_1, ν_2 the IMEX scheme will be stable. Computations were performed using finite element spaces consisting of continuous piece-wise polynomials of degree 2. The code was implemented using the software package `FreeFEM++` [16].

5.1. Convergence rate study

Computational results are provided choosing parameters $\nu_1 = \nu_2 = 1, \kappa = 0.01, 1, 2, 4$. In the following tables, the norm $\|\mathbf{u}\|$ is the discrete $L^2(0,T;H^1(\Omega_1)\times H^1(\Omega_2))$ norm, given by

$$\|\mathbf{u}\| = \left(\sum_{n=1}^{N} \Delta t \left\{ |u_1(t_n)|^2_{H^1(\Omega_1)} + |u_2(t_n)|^2_{H^1(\Omega_2)} \right\}\right)^{1/2}, \qquad (5.1)$$

and $\|\mathbf{u}\|_I$ is the discrete $L^2(0,T;L^2(I))$ norm, given by

$$\|\mathbf{u}\|_I = \left(\sum_{n=1}^{N} \Delta t \|\mathbf{u}(t_n)\|^2_{L^2(I)}\right)^{1/2}, \qquad (5.2)$$

where $N = T/\Delta t$.

FIRST h	**SUBSTEP** Δt	$\|\mathbf{u}(t_n)-\mathbf{u}^n\|$	rate	$\|\mathbf{u}(t_n)-\mathbf{u}^n\|_I$	rate
1/2	1/2	2.39459e+0		1.89181e-1	
1/4	1/4	7.95856e-1	1.59	1.01190e-1	0.90
1/8	1/8	2.47129e-1	1.69	5.21486e-2	0.96
1/16	1/16	8.79092e-2	1.49	2.65604e-2	0.97
1/32	1/32	3.78155e-2	1.23	1.34294e-2	0.98
1/64	1/64	1.80809e-2	1.06	6.75763e-3	0.99
SECOND h	**SUBSTEP** Δt	$\|\mathbf{u}(t_n)-\mathbf{u}^n\|$	rate	$\|\mathbf{u}(t_n)-\mathbf{u}^n\|_I$	rate
1/2	1/2	2.36605e+0		1.30618e-1	
1/4	1/4	7.54202e-1	1.65	2.25005e-2	2.54
1/8	1/8	2.06455e-1	1.87	6.22376e-3	1.85
1/16	1/16	5.36284e-2	1.94	1.74183e-3	1.84
1/32	1/32	1.36432e-2	1.97	4.73823e-4	1.88
1/64	1/64	3.43948e-3	1.99	1.25325e-4	1.92

Table 1: Errors for computed approximations, $\kappa = 0.01$

Tables 1 - 4 give the errors produced using Algorithm 2.3 with $\kappa = 0.01, 1, 2, 4$, respectively. The errors are calculated in the norms (5.1),(5.2) in all cases, for both uncorrected and corrected substeps. For each spatial mesh size h the time step size is chosen to be $\Delta t = h$. The errors should then scale proportional to $\Delta t + h^2 = O(h)$ for the uncorrected substeps and $\Delta t^2 + h^2 = O(h^2)$ for the corrected substeps.

Convergence of the IMEX scheme (uncorrected substeps) and Algorithm 2.3 is clear for $\kappa = 0.1, 1, 2$. The uncorrected substeps show first order convergence in h, while the corrected substeps show very nearly second order convergence in the norm (5.1), consistent with the theory. When $\kappa = 4$ the theory predicts a time step restriction $\Delta t \leq \frac{C}{\kappa^2} = \frac{C}{16}$, explaining the lack of convergence at larger mesh sizes choosing $h = \Delta t$ in Table 4.

FIRST	**SUBSTEP**				
h	Δt	$\|\mathbf{u}(t_n) - \mathbf{u}^n\|$	rate	$\|\mathbf{u}(t_n) - \mathbf{u}^n\|_I$	rate
1/2	1/2	6.66799e-2		1.87087e-2	
1/4	1/4	2.49435e-2	1.42	8.73540e-3	1.10
1/8	1/8	9.28931e-3	1.43	4.07453e-3	1.10
1/16	1/16	3.98260e-3	1.22	1.97159e-3	1.05
1/32	1/32	1.88625e-3	1.08	9.73713e-4	1.02
1/64	1/64	9.28461e-4	1.02	4.84781e-4	1.01
SECOND	**SUBSTEP**				
h	Δt	$\|\mathbf{u}(t_n) - \mathbf{u}^n\|$	rate	$\|\mathbf{u}(t_n) - \mathbf{u}^n\|_I$	rate
1/2	1/2	5.74386e-2		5.78786e-3	
1/4	1/4	1.92132e-2	1.58	1.92903e-3	1.59
1/8	1/8	5.33777e-3	1.85	5.18933e-4	1.89
1/16	1/16	1.40746e-3	1.92	1.38139e-4	1.91
1/32	1/32	3.66342e-4	1.94	3.75883e-5	1.88
1/64	1/64	9.57063e-5	1.94	1.03896e-5	1.86

Table 2: Errors for computed approximations, $\kappa = 1$

FIRST	**SUBSTEP**				
h	Δt	$\|\mathbf{u}(t_n) - \mathbf{u}^n\|$	rate	$\|\mathbf{u}(t_n) - \mathbf{u}^n\|_I$	rate
1/2	1/2	5.97678e-2		2.05628e-2	
1/4	1/4	2.33065e-2	1.36	9.46651e-3	1.12
1/8	1/8	7.91217e-3	1.56	3.52150e-3	1.43
1/16	1/16	3.15609e-3	1.33	1.54336e-3	1.19
1/32	1/32	1.45880e-3	1.11	7.45606e-4	1.05
1/64	1/64	7.12773e-4	1.03	3.69198e-4	1.01
SECOND	**SUBSTEP**				
h	Δt	$\|\mathbf{u}(t_n) - \mathbf{u}^n\|$	rate	$\|\mathbf{u}(t_n) - \mathbf{u}^n\|_I$	rate
1/2	1/2	4.73283e-2		6.29809e-3	
1/4	1/4	1.68062e-2	1.49	3.73213e-3	0.75
1/8	1/8	4.42209e-3	1.93	5.94709e-4	2.65
1/16	1/16	1.13583e-3	1.96	4.90307e-5	3.60
1/32	1/32	2.96786e-4	1.94	1.98341e-5	1.31
1/64	1/64	7.80296e-5	1.93	6.89075e-6	1.53

Table 3: Errors for computed approximations, $\kappa = 2$

The continuity of the trace operator from $H^1(\Omega_i)$ to $L^2(\partial\Omega_i)$, $i = 1, 2$, guarantees that the asymptotic rate of convergence in the norm (5.2) is at least as high as using the norm (5.1). This is observed for smaller values of κ as in Tables 1-2. It was observed that as the value of κ increases there is a tendency for the pointwise errors to concentrate near the interface, which explains the smaller convergence rates in the norm (5.2). This is particularly revealing in the case $\kappa = 4$ and $h = 1/16$, where the largest reported errors occur, shown in Figure 2. As detailed in the analysis of

FIRST h	SUBSTEP Δt	$\|\mathbf{u}(t_n) - \mathbf{u}^n\|$	rate	$\|\mathbf{u}(t_n) - \mathbf{u}^n\|_I$	rate
1/2	1/2	8.94746e-2		4.43531e-2	
1/4	1/4	1.79106e-1	—-	9.93188e-2	—-
1/8	1/8	7.73654e-1	—-	4.20286e-1	—-
1/16	1/16	2.80587e+0	—-	1.43856e+0	—-
1/32	1/32	1.63780e-1	4.10	7.53047e-2	4.26
1/64	1/64	4.73818e-4	8.43	2.28790e-4	8.36
SECOND h	**SUBSTEP** Δt	$\|\mathbf{u}(t_n) - \mathbf{u}^n\|$	rate	$\|\mathbf{u}(t_n) - \mathbf{u}^n\|_I$	rate
1/2	1/2	1.98918e-1		1.07760e-1	
1/4	1/4	8.57831e-1	—-	4.78724e-1	—-
1/8	1/8	6.78714e+0	—-	3.70902e+0	—-
1/16	1/16	41.47450e+0	—-	21.42870e+0	—-
1/32	1/32	3.52984e+0	3.55	1.63776e+0	3.71
1/64	1/64	1.75157e-4	14.30	7.14602e-5	14.48

Table 4: Errors for computed approximations, $\kappa = 4$

[11], when κ is larger than $\min\{\nu_1, \nu_2\}$ the base IMEX scheme has a time step restriction for stability. The correction step depends upon this initial IMEX step and thus the second order convergence rate is not observed when $\kappa = 4$ for the coarse time step sizes shown here.

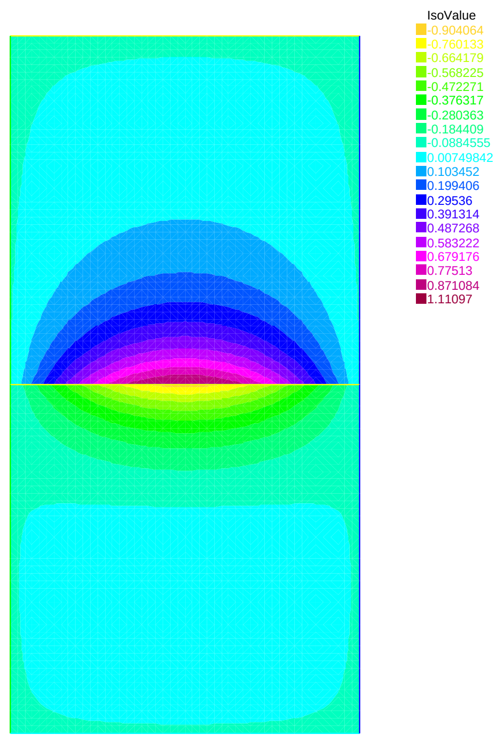

Figure 2: Interpolated SISDC errors at T=1, $\kappa = 4$ and $h = 1/16$.

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