Problem 2

In the given family of functions is the general solution of the differential equation on the indicated interval.

Find a member of the family that is a solution of the initial value problem.

\( y = c_1 e^{4x} + c_2 e^{-x} \) \((-\infty, \infty)\)

\( y'' - 3y' - 4y = 0, \ y(0) = 1, \ y'(0) = 2 \)

\( y = c_1 e^{4x} + c_2 e^{-x} \) (1)

\( y(0) = c_1 e^{4(0)} + c_2 e^{-0} \)

Since \( e^{0} = 1 \) and \( y(0) = 1 \)

\( y(0) = c_1 + c_2 = 1 \)

\( c_2 = 1 - c_1 \) (2)

\( y' = 4c_1 e^{4x} - c_2 e^{-x} \)

\( y'(0) = 4c_1 e^{4(0)} - c_2 e^{-0} \)

Since \( e^{0} = 1 \) and \( y'(0) = 2 \)

\( y'(0) = 4c_1 - c_2 = 2 \)

\( c_2 = 4c_1 - 2 \) (3)

Combine (2) and (3)

\( 1 - c_1 = 4c_1 - 2 \)

\( 5c_1 = 3 \)

\( c_1 = \frac{3}{5} \)

Substitute \( c_1 \) into (2)

\( c_2 = 1 - \frac{3}{5} \)

\( c_2 = \frac{2}{5} \)

Substitute \( c_1 \) and \( c_2 \) into (1)

\( y = \frac{3}{5} e^{4x} + \frac{2}{5} e^{-x} \)

Does the solution fit

\( y'' - 3y' - 4y = 0 \)

\( y = \frac{3}{5} e^{4x} + \frac{2}{5} e^{-x} \)

\( y' = \frac{12}{5} e^{4x} - \frac{2}{5} e^{-x} \)

\( y'' = \frac{48}{5} e^{4x} + \frac{2}{5} e^{-x} \)

\( \frac{48}{5} e^{4x} + \frac{2}{5} e^{-x} - 3 \left( \frac{12}{5} e^{4x} - \frac{2}{5} e^{-x} \right) - 4 \left( \frac{3}{5} e^{4x} + \frac{2}{5} e^{-x} \right) \)

\( \left( \frac{48}{5} e^{4x} - \frac{36}{5} e^{4x} - \frac{12}{5} e^{4x} \right) + \left( \frac{2}{5} e^{-x} + \frac{6}{5} e^{-x} - \frac{8}{5} e^{-x} \right) = 0 \)

Is a solution.
Determine whether the given set of functions is linearly independent on the interval \((-\infty, \infty)\):

\[ f_1(x) = 1 + x \quad f_2(x) = x \quad f_3(x) = x^2 \]

Suppose

\[ c_1(1 + x) + c_2(x) + c_3(x^2) = 0 \]

Take \(c_1 = 0\), \((c_1 + c_2) = 0\) and \(c_3 = 0\).

Since \(c_1 = 0\), \(c_2 = 0\).

All \(c\)'s equal zero.

\[ \therefore \text{Functions are linearly independent} \]

OR

Using the Wronskian (Definition 4.1.2 & Theorem 4.1.3)

\[
W(f_1, f_2, \ldots, f_n) = \begin{vmatrix}
    f_1 & f_2 & \cdots & f_n \\
    f_1' & f_2' & \cdots & f_n' \\
    \vdots & \vdots & \ddots & \vdots \\
    f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)}
\end{vmatrix}
\]

\[
W(1 + x, x, x^2) = \begin{vmatrix}
    1 + x & x & x^2 \\
    1 & 1 & 2x \\
    0 & 0 & 2
\end{vmatrix}
\]

\[
= (1+x)[(1)(2) - (2x)(0)] - (x)[(1)(2) - (2x)(0)] + x^2[(1)(0) - (1)(0)]
\]

\[
= 2 + 2x - 2x = 2
\]

\[ W(1 + x, x, x^2) = 2 \]

Since \(W(1 + x, x, x^2) \neq 0\),

the given functions are linearly independent.
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Solutions

Verify that the given functions form a fundamental set of solutions of the differential equation on the indicated interval.

Form the general solution
\[ 4y'' - 4y' + y = 0 \quad \text{for} \quad e^{x/2}, \quad xe^{x/2}, \quad (-\infty, \infty) \]

Verify that the functions are linearly independent using the Wronskian

\[ W(e^{x/2}, xe^{x/2}) = \begin{vmatrix} e^{x/2} & xe^{x/2} \\ \frac{1}{2} e^{x/2} & \frac{1}{2}xe^{x/2} + e^{x/2} \end{vmatrix} \]

\[ = \frac{1}{2}xe^{x/2} - \frac{1}{2}xe^{x/2} + e^{x/2} = e^{x} \neq 0 \]

Linearly independent

Solve for \( e^{x/2} \) term

\[ Y_1 = e^{x/2} \]
\[ Y_1' = \frac{1}{2} e^{x/2} \]
\[ Y_1'' = \frac{1}{4} e^{x/2} \]
\[ 4(\frac{1}{4} e^{x/2}) - 4(\frac{1}{2} e^{x/2}) + e^{x} = 0 \]
\[ 0 = 0 \]

\( e^{x/2} \) is a solution

The general solution is
\[ y = C_1 e^{x/2} + C_2 xe^{x/2} \]
Verify that the given two-parameter family of functions is the general solution of the general solution of the nonhomogeneous differential equation on the indicated interval.

\[ y'' - 7y' + 10y = 24e^x \]
\[ y = c_1 e^{2x} + c_2 e^{5x} + 6e^x, \quad (-\infty, \infty) \]

Find the first and second derivative
\[ y' = 2c_1 e^{2x} + 5c_2 e^{5x} + 6e^x \]
\[ y'' = 4c_1 e^{2x} + 25c_2 e^{5x} + 6e^x \]

Substitute \( y, y', \) and \( y'' \)
\[ 4c_1 e^{2x} + 25c_2 e^{5x} + 6e^x - 7(2c_1 e^{2x} + 5c_2 e^{5x} + 6e^x) \]
\[ + 10(c_1 e^{2x} + c_2 e^{5x} + 6e^x) \]
\[ (4c_1 e^{2x} - 14c_1 e^{2x} + 10c_1 e^{2x}) + (25c_2 e^{5x} - 35c_2 e^{5x} + 10c_2 e^{5x}) \]
\[ + 6e^x - 42e^x + 60e^x = 24e^x \]

\[ y = c_1 e^{2x} + c_2 e^{5x} + 6e^x \] is a solution for the nonhomogeneous differential equation, where \( y_1 = e^{2x} \) and \( y_2 = e^{5x} \) form a fundamental set of solutions and \( y_p = 6e^x \) is a particular solution.