Find the general solution of the given 2nd-order differential equation:

\[ y'' - y' - 6y = 0 \]

The auxiliary equation is

\[ m^2 - m - 6 = 0 \]

Solve by factoring

\[(m - 3)(m + 2)\]

\[ m_1 = 3, \ m_2 = -2 \]

\[ m_1 \text{ and } m_2 \text{ are real and distinct} \]

The solution takes the form

\[ y = C_1 e^{3x} + C_2 e^{-2x} \]

\[ y = C_1 e^{3x} + C_2 e^{-2x} \]
Problem 15

Find the general solution of the given higher-order differential equation.

\[ y'' - 4y' - 5y = 0 \]

The 2nd degree polynomial is

\[ m^2 - 4m - 5 = 0 \]
\[ m (m^2 - 4m - 5) = 0 \]
\[ m ((m - 5)(m + 1)) = 0 \]

\[ m_1 = 0 \text{, } m_2 = -1 \text{, } m_3 = 5 \]

Since all roots are real and distinct

\[ y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x} \]

\[ y = c_1 e^{0x} + c_2 e^{-x} + c_3 e^{5x} \]

\[ e^0 = 1 \]

\[ y = c_1 + c_2 e^{-x} + c_3 e^{5x} \]
Problem 23

Find the general solution of the given higher-order differential equation.

\[ y^{(m)} + y^{(m-1)} + y = 0 \]

The \( m \)-th order polynomial is

\[ m^4 + m^3 + m^2 \]

\[ m^2 (m^2 + m + 1) = 0 \]

\[ m_1 = 0, \ m_2 = 0 \]

Solve for \( m^2 + m + 1 \) using the quadratic equation

\[ m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{-3}}{2} \]

\[ m_3 = -\frac{1}{2} + \frac{\sqrt{3}}{2} \ i, \quad m_4 = -\frac{1}{2} - \frac{\sqrt{3}}{2} \ i \]

The real part of the solution with repeated roots is,

\[ y = c_1 e^{m_1 x} + c_2 x e^{m_1 x} \]

\[ y = c_1 e^{0x} + c_2 x e^{0x}, \text{ where } e^0 = 1 \]

\[ y = c_1 + c_2 x \]

The complex part of the solution is

\[ y = e^{\omega x}(c_3 \cos \beta x + c_4 \sin \beta x) \]

where \( m_3 = \alpha + i\beta, \ m_4 = \alpha - i\beta \)

\[ \alpha = -\frac{1}{2}, \ \beta = \frac{\sqrt{3}}{2} \]

\[ y = e^{\frac{x}{2}}(c_3 \cos \left(\frac{\sqrt{3}}{2} x\right) + c_4 \sin \left(\frac{\sqrt{3}}{2} x\right)) \]

Combine the real solution with the complex solution

\[ y = c_1 + c_2 x + e^{\frac{x}{2}}(c_3 \cos \left(\frac{\sqrt{3}}{2} x\right) + c_4 \sin \left(\frac{\sqrt{3}}{2} x\right)) \]
Solve the given boundary-value problem
\[ y'' + 4y = 0 \quad y(0) = 0 \quad y(\pi) = 0 \]

The auxiliary equation is
\[ m^2 + 4 = 0 \]
\[ m = \pm 2i \]

If \( m = \alpha \pm i\beta \) where \( \alpha = 0 \) the solution takes the form of
\[
y = C_1 \cos \beta x + C_2 \sin \beta x
\]
\[
y = C_1 \cos 2x + C_2 \sin 2x
\]

Knowing that \( y(0) = 0 \) and \( y(\pi) = 0 \)
\[
\sin(0) & \sin(\pi) = 0
\]
\[
\cos(0) & \cos(\pi) = 1
\]
This holds if \( x = 2\pi, 3\pi, 4\pi, \ldots \)

\[
\therefore \quad y = C_1(1) + C_2(0) = 0
\]
\[
C_1 = 0
\]

The solution is
\[
y = C_2 \sin 2x
\]